

New Deep Holes of Generalized Reed-Solomon Codes

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Abstract

Deep holes play an important role in the decoding of generalized Reed-Solomon codes. Recently, Wu and Hong [11] found a new class of deep holes for standard Reed-Solomon codes. In the present paper, we give a concise method to obtain a new class of deep holes for generalized Reed-Solomon codes. In particular, for standard Reed-Solomon codes, we get the new class of deep holes given in [11].

Li and Wan [6] studied deep holes of generalized Reed-Solomon codes $GRS_k(\mathbb{F}_q, D)$ and characterized deep holes defined by polynomials of degree $k + 1$. They showed that this problem is reduced to be a subset sum problem in finite fields. Using the method of Li and Wan, we obtain some new deep holes for special Reed-Solomon codes over finite fields with even characteristic. Furthermore, we study deep holes of the extended Reed-Solomon code, i.e., $D = \mathbb{F}_q$ and show polynomials of degree $k + 2$ can not define deep holes.

keywords: coding theory, Reed-Solomon codes, list decoding, deep holes, multiplicative character, quadratic equation.

1 Introduction

Let \mathbb{F}_q be the finite field of q elements with characteristic p . Fix a subset $D = \{x_1, \dots, x_n\} \subseteq \mathbb{F}_q$, which is called the evaluation set. The generalized

Reed-Solomon code $C = GRS_k(\mathbb{F}_q, D)$ of length n and dimension k over \mathbb{F}_q is defined to be

$$GRS_k(\mathbb{F}_q, D) = \{(f(x_1), \dots, f(x_n)) \in \mathbb{F}_q^n \mid f(x) \in \mathbb{F}_q[x], \deg f(x) \leq k-1\}.$$

Its elements are called codewords. The most widely used cases are $D = \mathbb{F}_q$ or \mathbb{F}_q^* . For the case $D = \mathbb{F}_q^*$, it is just the standard Reed-Solomon code. For the case $D = \mathbb{F}_q$, it is the extended Reed-Solomon code.

For any word $u \in \mathbb{F}_q^n$, by the Lagrange interpolation, there is a polynomial f of degree $\leq n-1$ such that

$$u = u_f = (f(x_1), f(x_2), \dots, f(x_n)).$$

Clearly, $u_f \in GRS_k(\mathbb{F}_q, D)$ if and only if $\deg(f) \leq k-1$. We also say that u_f is defined by the polynomial $f(x)$.

Let C be an $[n, k, d]$ linear code over \mathbb{F}_q . The *error distance* of any word $u \in \mathbb{F}_q^n$ to C is defined to be

$$d(u, C) = \min\{d(u, v) \mid v \in C\}.$$

where

$$d(u, v) = \#\{i \mid u_i \neq v_i, 1 \leq i \leq n\}$$

is the Hamming distance between words u and v . The error distance play an important role in the list decoding of Reed-Solomon codes. Given a received word $u \in \mathbb{F}_q^n$, if the error distance is small, say, $d(u, C) \leq n - \sqrt{nk}$, then the list decoding algorithm of Sudan [10] and Guruswami-Sudan [4] provides a polynomial time algorithm for the decoding of u . When the error distance increases, the maximal likelihood decoding becomes more complicated, in fact, **NP**-complete for generalized Reed-Solomon codes [5].

The most important algorithmic problem in coding theory is the maximal likelihood decoding: given a word $u \in \mathbb{F}_q^n$, find a codeword $v \in C$ such that $d(u, v) = d(u, C)$. The decision version of this problem is essentially computing the error distance $d(u, C)$ for a received word u . For the generalized Reed-Solomon codes $GRS_k(\mathbb{F}_q, D)$, the following result is well-known.

Lemma 1.1 ([6]). *For any $k \leq \deg(f) \leq n-1$, we have*

$$n - \deg(f) \leq d(u_f, C) \leq n - k.$$

In particular, the word u is called a *deep hole* if the above upper bound is attained, i.e., $d(u, C) = n - k$. Following Lemma 1.1, we see that the vectors defined by polynomials of degree k are deep holes.

For the standard Reed-Solomon code $GRS_k(\mathbb{F}_q, \mathbb{F}_q^*)$, based on numerical computations, Cheng and Murray [2] conjectured that vectors defined by polynomials of degree k are the only deep holes possible. As a theoretical evidence, they proved that their conjecture is true for words u_f defined by polynomial f if $d = \deg(u_f) - k$ is small and q is sufficiently large compared to $d + k$. For those words defined by polynomials in $\mathbb{F}_q[x]$ of low degrees, Li and Wan [8] applied the method of Cheng and Wan [3] to study the error distance $d(u, C)$ for

the standard Reed-Solomon code. Liao [9] extended the results in [8] to those words defined by polynomials in $\mathbb{F}_q[x]$ of high degrees. Recently, by means of a deeper study of the geometry of hypersurfaces, Cafure and et al. [1] made some improvement of the results in [8].

Recently, Wu and Hong [11] found a new class of deep holes for standard Reed-Solomon codes. They considered the standard Reed-Solomon code $GRS_k(\mathbb{F}_q, \mathbb{F}_q^*)$ as a cyclic code with the generator polynomial

$$g(x) = (x - \alpha)(x - \alpha^2) \cdots (x - \alpha^{n-k}),$$

where α is a primitive element of \mathbb{F}_q . They found two classes of deep holes, namely $a \frac{g(x)}{x - \alpha} + l(x)g(x)$ and $a \frac{g(x)}{x - \alpha^{n-k}} + l(x)g(x)$ for any $a \in \mathbb{F}_q^*$ and $l(x) \in \mathbb{F}_q[x]$ with $\deg(l(x)) \leq k - 1$. After the inverse discrete Fourier transform, the latter is just the trivial ones defined by polynomials of degree k , but the former is new. The new deep holes are of the form $ux^{q-2} + l(x)$ for any $u \neq 0$ and $l(x) \in \mathbb{F}_q[x]$ with $\deg(l(x)) \leq k - 1$. And then they conjectured that there are no more deep holes for standard Reed-Solomon codes.

In Section 2, we study deep holes of generalized Reed-Solomon codes. Using a simple method, we give a new class of deep holes for any generalized Reed-Solomon code in the case $D \neq \mathbb{F}_q$. In Section 3, using the explicit formula given in [8], we obtain new deep holes for special Reed-Solomon codes over finite fields with even characteristic, which are not contained in the two classes of deep holes given in Section 2. In Section 4, we study deep holes of extended Reed-Solomon codes, and characterize the deep holes defined by polynomials of degree $k + 2$. We show that the existence of these deep holes is equivalent to non-existence of solutions for a class of quadratic equations. We prove the quadratic equation always has solutions. Then there are no deep holes defined by polynomials of degree $k + 2$.

2 A New class of Deep Holes for Generalized Reed-Solomon Codes

In this section, we consider generalized Reed-Solomon codes $GRS_k(\mathbb{F}_q, D)$ over the finite field \mathbb{F}_q . We give a new class of deep holes for $GRS_k(\mathbb{F}_q, D)$ besides the trivial ones defined by the polynomials of degree k .

In the case that $D = \mathbb{F}_q^*$, i.e., standard Reed-Solomon codes, Wu and Hong considered them as cyclic codes and gave a new class of deep holes defined by polynomials $f = ax^{q-2} + f_{\leq k-1}(x)$ ($a \neq 0$) where $f_{\leq k-1}(x)$ represents the terms of degree $\leq k - 1$ in the polynomial f . But for the general valuation set $D \neq \mathbb{F}_q^*$, the generalized Reed-Solomon code $GRS_k(\mathbb{F}_q, D)$ can not be considered as a cyclic code, so their method is invalid. Below we give a simple method to show that for any $b \notin D$, these words defined by polynomials $f = a(x - b)^{q-2} + f_{\leq k-1}(x)$ ($a \neq 0$) are still deep holes for generalized Reed-Solomon codes $GRS_k(\mathbb{F}_q, D)$.

First, we give the key lemma.

Lemma 2.1. *Let \mathbb{F}_q be a finite field with q elements and $\mathbb{F}_q[x]$ the ring of polynomials over \mathbb{F}_q . For any $D \subsetneq \mathbb{F}_q$, any $b \notin D$, and any polynomial $(x-b)^{q-2} - g(x) \in \mathbb{F}_q[x]$ with $\deg(g(x)) \leq k-1$, $a \in D$ is a zero of $(x-b)^{q-2} - g(x)$ if and only if a is a zero of $1 - (x-b)g(x)$. So the polynomial $(x-b)^{q-2} - g(x)$ has at most k zeros in D .*

Proof. Notice that the polynomial $(x-b)^{q-2} - g(x)$ has the same zeros as the polynomial $(x-b)^2((x-b)^{q-2} - g(x)) = (x-b) - (x-b)^2g(x)$ on D for $b \notin D$. While the latter has the same zeros as the polynomial $1 - (x-b)g(x)$ on D . So the polynomial $(x-b)^{q-2} - g(x)$ has the same zeros as the polynomial $1 - (x-b)g(x)$ on D . But the degree of $1 - (x-b)g(x)$ is not larger than k , so all the mentioned polynomials has at most k zeros in D . \square

By this lemma, we easily obtain a new class of deep holes for $GRS_k(\mathbb{F}_q, D)$ besides the trivial ones defined by polynomials of degree k .

Theorem 2.2. *Let \mathbb{F}_q be a finite field with q elements and $C = GRS_k(\mathbb{F}_q, D)$ the generalized Reed-Solomon code over \mathbb{F}_q , where $D \subsetneq \mathbb{F}_q$ is the evaluation set with cardinality n . Then for any $b \notin D$, polynomials $f = a(x-b)^{q-2} + f_{\leq k-1}(x)$ ($a \neq 0$) define deep holes for $GRS_k(\mathbb{F}_q, D)$.*

Proof. Without loss of generality, we assume $D \subseteq \mathbb{F}_q^*$, $b = 0$, and we may only consider the polynomial $f = x^{q-2}$. By Lemma 2.1, for any polynomial $g(x) \in \mathbb{F}_q[x]$ with $\deg(g(x)) \leq k-1$, the polynomial $f - g$ has at most k zeros in D . So the error distance

$$d(u_f, C) = n - \max_{g \in \mathbb{F}_q[x], \deg(g) \leq k-1} |\{\text{zeros of } f - g\}| \geq n - k.$$

On the other hand, for any $S \subseteq D$ with $|S| = k$, set

$$g_S = \frac{1 - a_S \prod_{\gamma \in S} (x - \gamma)}{x}, \text{ where } a_S = (-1)^k \prod_{\gamma \in S} \gamma^{-1}.$$

Then $g_S \in \mathbb{F}_q[x]$ has degree $k-1$ and the polynomial

$$1 - xg_S(x) = a_S \prod_{\gamma \in S} (x - \gamma)$$

has k zeros in D . From the argument in the proof of Lemma 2.1, the polynomial $f - g_S = x^{q-2} - g_S$ has k zeros in D . So the lower bound for $d(u_f, C)$ can be attained, i.e.,

$$d(u_f, C) = n - k.$$

In other words, the polynomial $f = x^{q-2}$ defines a deep hole for the generalized Reed-Solomon code $C = GRS_k(\mathbb{F}_q, D)$. \square

In particular, if we take $D = \mathbb{F}_q^*$, then by Theorem 2.2 the polynomials $f = ax^{q-2} + f_{\leq k-1}(x)$ ($a \neq 0$) define deep holes for the standard Reed-Solomon code $GRS_k(\mathbb{F}_q, \mathbb{F}_q^*)$, which were given by Wu and Hong [11].

3 A New class of Deep Holes for Special Reed-Solomon Codes over Finite Fields with Even Characteristic

In this section, we consider Reed-Solomon codes $GRS_k(\mathbb{F}_q, D)$ over the finite field \mathbb{F}_q with even characteristic. And we give a new class of deep holes for some Reed-Solomon codes in this case. Li and Wan [8] characterized the deep holes defined by polynomials of degree $k+1$ as follows.

Let $f = x^{k+1} - bx^k + f_{\leq k-1}(x) \in \mathbb{F}_q[x]$. By Lemma 1.1, $n - k - 1 \leq d(u_f, C) \leq n - k$. If f does not define a deep hole, i.e., $d(u_f, C) = n - k - 1$, it is equivalent to that there exists $g \in \mathbb{F}_q[x]$, $\deg(g) \leq k - 1$ such that

$$x^{k+1} - bx^k + f_{\leq k-1}(x) - g(x) = \prod_{j=1}^{k+1} (x - a_{i_j})$$

for a $(k+1)$ -subset $\{a_{i_1}, \dots, a_{i_{k+1}}\} \subseteq D$. It is also equivalent to

$$a_{i_1} + \dots + a_{i_{k+1}} = b$$

for a $(k+1)$ -subset $\{a_{i_1}, \dots, a_{i_{k+1}}\} \subseteq D$.

Following the notation in [6], for any subset D of \mathbb{F}_q^* , let

$$N(t, b, D) = \#\{S \subseteq D \mid \sum_{x \in S} x = b, \#S = t\}.$$

Then the polynomial $f = x^{k+1} - bx^k + f_{\leq k-1}(x)$ defines a deep hole if and only if

$$N(t, b, D) = 0.$$

Taking $D = \mathbb{F}_q^*$ or $D = \mathbb{F}_q^* \setminus \{1\}$, Li and Wan presented the explicit formula for $N(t, b, D)$.

Proposition 3.1 ([6]). *Let \mathbb{F}_q be a finite field with q elements.*

(i) *For $D = \mathbb{F}_q^*$,*

$$N(t, 0, \mathbb{F}_q^*) = \frac{1}{q} \binom{q-1}{t} + (-1)^{t+[t/p]} \frac{q-1}{q} \binom{q/p-1}{[t/p]}.$$

(ii) *For $D = \mathbb{F}_q^* \setminus \{1\}$, let*

$$R_t = -p(-1)^{[t/p]} \binom{q/p-2}{[t/p]} + (p-1 - \langle t \rangle_p) (-1)^{[t/p]} \binom{q/p-1}{[t/p]}$$

and

$$M(t, b) = -(-1)^{[t/p]} \binom{q/p-2}{[t/p]} + \delta_{b,t} (-1)^{[t/p]} \binom{q/p-1}{[t/p]},$$

where $\langle t \rangle_p$ denotes the least non-negative residue of t modulo p , and $\delta_{b,t} = 1$ if $\langle b \rangle_p$ is greater than $\langle t \rangle_p$ and $\delta_{b,t} = 0$ otherwise. Then

$$N(t, b, D) = \frac{1}{q} \binom{q-2}{t} + \frac{1}{q} R_t - (-1)^t M(t, t-b).$$

Thanks to the explicit formulae, the following is immediate for the case that the characteristic $p = 2$ and $q > 4$. For $D = \mathbb{F}_q^*$ or $D = \mathbb{F}_q^* \setminus \{1\}$,

$$N(q-3, 0, D) = 0.$$

Then $d(u_f, C) = n - k$. Therefore, we obtain a class of new deep holes for the Reed-Solomon code $GRS_{q-4}(\mathbb{F}_q, D)$.

Theorem 3.2. *Let \mathbb{F}_q be a finite field with characteristic 2 and $D = \mathbb{F}_q^*$ or $D = \mathbb{F}_q^* \setminus \{1\}$. If $q > 4$, then the vectors defined by polynomials $ax^{q-3} + f_{\leq k-1}(x)$ ($a \neq 0$) are deep holes for the Reed-Solomon code $GRS_{q-4}(\mathbb{F}_q, D)$.*

Wu and Hong conjectured that there are only two classes of deep holes for the standard Reed-Solomon code $GRS_k(\mathbb{F}_q, \mathbb{F}_q^*)$ defined by polynomials of the form $ax^k + f_{\leq k-1}(x)$ and $ax^{q-2} + f_{\leq k-1}(x)$ ($a \neq 0$). While in case for the special standard Reed-Solomon code $GRS_{q-4}(\mathbb{F}_q, \mathbb{F}_q^*)$ over the finite field \mathbb{F}_q with even characteristic, Theorem 3.2 gives a counterexample of the conjecture of Wu and Hong [11], i.e., deep holes defined by polynomials $ax^{q-3} + f_{\leq k-1}(x)$ ($a \neq 0$).

4 Deep Holes defined by Polynomials of Degree $k + 2$

In this section, we consider the extended Reed-Solomon code $GRS_k(\mathbb{F}_q, \mathbb{F}_q)$ over the finite field \mathbb{F}_q of $q > 5$ elements with odd characteristic. For the polynomial of the form $f = x^{k+2} - ax^{k+1} - bx^k + f_{\leq k-1}(x)$, by Lemma 1.1, we have $n - k - 2 \leq d(u_f, C) \leq n - k$. Then $d(u_f, C) \leq n - k - 1$, i.e., such f does not define a deep hole, if and only if there exists some polynomial $g \in \mathbb{F}_q[x]$ of degree $\deg(g) \leq k - 1$ such that

$$f - g = (x - \gamma) \prod_{\beta \in S} (x - \beta),$$

for some $S \subseteq \mathbb{F}_q$ with cardinality $k + 1$, and some $\gamma \in \mathbb{F}_q$. This implies that

$$\begin{cases} a &= \gamma + \sum_{\beta \in S} \beta, \\ b &= -\sum_{\{\beta_1, \beta_2\} \subset S} \beta_1 \beta_2 - \gamma \sum_{\beta \in S} \beta. \end{cases}$$

Hence we have the following lemma.

Lemma 4.1. *The polynomial $f = x^{k+2} - ax^{k+1} - bx^k + f_{\leq k-1}(x) \in \mathbb{F}_q[x]$ can not define a deep hole if and only if there are some $S \subseteq \mathbb{F}_q$ with cardinality $k + 1$, and some $\gamma \in \mathbb{F}_q$ such that*

$$\begin{cases} a &= \gamma + \sum_{\beta \in S} \beta, \\ b &= -\sum_{\{\beta_1, \beta_2\} \subset S} \beta_1 \beta_2 - \gamma \sum_{\beta \in S} \beta. \end{cases}$$

From the two equalities, we have

$$b = - \sum_{\{\beta_1, \beta_2\} \subset S} \beta_1 \beta_2 - (a - \sum_{\beta \in S} \beta) \sum_{\beta \in S} \beta.$$

Hence, to search all such f that does not define a deep hole, we need to find all $a \in \mathbb{F}_q$ and $b \in \mathbb{F}_q$ such that the equation

$$\begin{cases} b = - \sum_{1 \leq i < j \leq k+1} X_i X_j + (\sum_{i=1}^{k+1} X_i - a) \sum_{i=1}^{k+1} X_i, \\ X_i \neq X_j, \quad \text{for all } i \neq j, \end{cases}$$

i.e.,

$$\begin{cases} b = \sum_{1 \leq i < j \leq k+1} X_i X_j + \sum_{i=1}^{k+1} X_i^2 - a \sum_{i=1}^{k+1} X_i, \\ X_i \neq X_j, \quad \text{for all } i \neq j \end{cases}$$

has solutions in \mathbb{F}_q .

Indeed, we will show that the equation always has solutions for any $a, b \in \mathbb{F}_q$. Let $t = k + 1$, so we have

Theorem 4.2. *For any $a, b \in \mathbb{F}_q$, for all $3 \leq t \leq q - 2$, the equation*

$$\begin{cases} b = \sum_{1 \leq i < j \leq t} X_i X_j + \sum_{i=1}^t X_i^2 - a \sum_{i=1}^t X_i, \\ X_i \neq X_j, \quad \text{for all } i \neq j \end{cases}$$

has solutions in \mathbb{F}_q .

The proof we give is highly nontrivial, so we present the proof in the Appendix.

Since $\deg(f) = k + 2 \leq n - 1 = q - 1$, we only consider $k \leq q - 3$. By Theorem 4.2, we obtain the main result of this section.

Theorem 4.3. *Let \mathbb{F}_q be a finite field of $q > 5$ elements and $GRS_k(\mathbb{F}_q, \mathbb{F}_q)$ the standard Reed-Solomon code. For all $2 \leq k \leq q - 3$, the polynomials $f = ux^{k+2} + ax^{k+1} + bx^k + f_{\leq k-1}(x) \in \mathbb{F}_q[x]$ ($u \in \mathbb{F}_q^*, a, b \in \mathbb{F}_q$) do not define deep holes.*

5 Conclusion

The result of Section 4 supports the conjecture of Cheng and Murray [2] for extended Reed-Solomon codes. And it is easy to see that as the evaluation set D becomes small, there will be more deep holes for the Reed-Solomon code $GRS_k(\mathbb{F}_q, D)$. In particular, in Section 2 we have seen that there is a new class of deep holes whenever there is an element $a \in \mathbb{F}_q \setminus D$. For finite fields \mathbb{F}_q with cardinality $q > 5$ and $D \subseteq \mathbb{F}_q$ with cardinality q or $q - 1$, Li and Wan [6] proved that there is no deep hole of $GRS_k(\mathbb{F}_q, D)$ defined by polynomials of degree $k + 1$ when $2 < k < q - 3$. In Section 4, we have seen that if the finite field \mathbb{F}_q has odd characteristic, there is also no deep hole of $GRS_k(\mathbb{F}_q, \mathbb{F}_q)$ defined by polynomials of degree $k + 2$ when $2 \leq k \leq q - 3$. Similarly, in this case, we can prove that there is also no deep hole of $GRS_k(\mathbb{F}_q, \mathbb{F}_q^*)$ defined by polynomials of degree $k + 2$ when $2 \leq k < q - 4$.

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A Appendix

To complete this paper, we give the proof of Theorem 4.2 in this section. We only prove the theorem for the case that $q = p > 2$ a prime integer. In the proof, we also need to assume $q \geq 257$. For prime $q < 257$, we can use the computer to check it.

For general prime power q , the proof is similar. Note that the reduction in Lemma A.1 is not valid when the characteristic $p \mid (t+1)$. But for small t (i.e., $t < c_1 q$ for some constant c_1), the method in the proof of Lemma A.3 (and in the part (b) of the proof for Theorem 4.2) still works; for large t (i.e., $t > c_2 q$ for some constant c_2), using Lemma A.2, it is enough to consider the complement set $\mathbb{F}_q \setminus \{X_1, \dots, X_t\}$. Together with the required version of Theorem A.7 (taking $c = 1/2$ and a proper ϵ in Theorem 5.3 in [7]), we can finish the proof of Theorem 4.2.

First, we give some lemmas. Due to the next lemma, it is reduced to the case $a = 0$.

Lemma A.1. *For any $a, b \in \mathbb{F}_q$, the equation*

$$\begin{cases} b &= \sum_{1 \leq i < j \leq t} X_i X_j + \sum_{i=1}^t X_i^2 - a \sum_{i=1}^t X_i, \\ X_i &\neq X_j, \quad \text{for all } i \neq j \end{cases}$$

has solutions in \mathbb{F}_q , if and only if for any $\beta \in \mathbb{F}_q$, the equation

$$\begin{cases} \beta &= \sum_{1 \leq i < j \leq t} X_i X_j + \sum_{i=1}^t X_i^2, \\ X_i &\neq X_j, \quad \text{for all } i \neq j \end{cases}$$

has solutions in \mathbb{F}_q .

Proof. The necessity part holds obviously.

For the other direction, for any $a, b \in \mathbb{F}_q$, taking $\beta = \frac{ta^2}{2(t+1)} + b$, suppose $(x_1, \dots, x_t) \in \mathbb{F}_q^t$ is a solution of the equation

$$\begin{cases} \beta &= \sum_{1 \leq i < j \leq t} X_i X_j + \sum_{i=1}^t X_i^2, \\ X_i &\neq X_j, \quad \text{for all } i \neq j. \end{cases}$$

Set $c = \frac{a}{t+1}$, then

$$\begin{aligned} & \sum_{1 \leq i < j \leq t} (x_i + c)(x_j + c) + \sum_{i=1}^t (x_i + c)^2 - a \sum_{i=1}^t (x_i + c) \\ &= \sum_{1 \leq i < j \leq t} x_i x_j + \sum_{i=1}^t x_i^2 + ((t+1)c - a) \sum_{i=1}^t x_i - tac + \frac{t(t+1)}{2} c^2 \\ &= b. \end{aligned}$$

So $(x_1 + c, \dots, x_t + c) \in \mathbb{F}_q^t$ is a solution of the equation

$$\begin{cases} b &= \sum_{1 \leq i < j \leq t} X_i X_j + \sum_{i=1}^t X_i^2 - a \sum_{i=1}^t X_i, \\ X_i &\neq X_j, \quad \text{for all } i \neq j. \end{cases}$$

□

Lemma A.2. For any $S \subset \mathbb{F}_q$ with cardinality $2 \leq |S| \leq q-2$, set $S' = \mathbb{F}_q \setminus S$. Then

$$\sum_{\{\beta_1, \beta_2\} \subset S} \beta_1 \beta_2 + \sum_{\beta \in S} \beta^2 = \sum_{\{\gamma_1, \gamma_2\} \subset S'} \gamma_1 \gamma_2.$$

Proof. We have

$$\begin{aligned} 0 &= \sum_{\{\beta_1, \beta_2\} \subset \mathbb{F}_q} \beta_1 \beta_2 + \sum_{\beta \in \mathbb{F}_q} \beta^2 \\ &= (\sum_{\{\beta_1, \beta_2\} \subset S} \beta_1 \beta_2 + \sum_{\beta \in S} \beta^2) + (\sum_{\{\gamma_1, \gamma_2\} \subset S'} \gamma_1 \gamma_2 + \sum_{\gamma \in S'} \gamma^2) \\ &\quad + \sum_{\beta \in S, \gamma \in S'} \beta \gamma \\ &= (\sum_{\{\beta_1, \beta_2\} \subset S} \beta_1 \beta_2 + \sum_{\beta \in S} \beta^2) + (\sum_{\{\gamma_1, \gamma_2\} \subset S'} \gamma_1 \gamma_2 + \sum_{\gamma \in S'} \gamma^2) \\ &\quad + (\sum_{\gamma \in S'} \gamma)(-\sum_{\gamma \in S'} \gamma) \\ &= \sum_{\{\beta_1, \beta_2\} \subset S} \beta_1 \beta_2 + \sum_{\beta \in S} \beta^2 - \sum_{\{\gamma_1, \gamma_2\} \subset S'} \gamma_1 \gamma_2. \end{aligned}$$

So

$$\sum_{\{\beta_1, \beta_2\} \subset S} \beta_1 \beta_2 + \sum_{\beta \in S} \beta^2 = \sum_{\{\gamma_1, \gamma_2\} \subset S'} \gamma_1 \gamma_2.$$

□

By this lemma, it is enough to consider the equation

$$\begin{cases} b = \sum_{1 \leq i < j \leq t} X_i X_j, \\ X_i \neq X_j, \quad \text{for all } i \neq j. \end{cases}$$

Lemma A.3. For any $b \in \mathbb{F}_q$ and $2 \leq t \leq \frac{q-1}{2}$, the equation

$$\begin{cases} b = \sum_{1 \leq i < j \leq t} X_i X_j, \\ X_i \neq X_j, \quad \text{for all } i \neq j \end{cases}$$

has solutions in \mathbb{F}_q .

Proof. We prove the statement by induction on t .

For $t = 2, 3$, one can easily check it.

Now we assume that the result is true for $t \geq 3$.

Then for $t + 1$, by induction hypothesis, suppose the equation

$$\sum_{1 \leq i < j \leq t} X_i X_j = b$$

has a solution $(x_1, \dots, x_t) \in \mathbb{F}_q^t$ such that $x_i \neq x_j$ for any $i \neq j$.

If $x_i \neq 0$ for all $1 \leq i \leq t$, then $(0, x_1, \dots, x_t) \in \mathbb{F}_q^{t+1}$ is a solution of the equation

$$\begin{cases} b = \sum_{1 \leq i < j \leq t+1} X_i X_j, \\ X_i \neq X_j, \quad \text{for all } i \neq j. \end{cases}$$

Then we finish the induction procedure.

Otherwise, without loss of generality, we assume $x_1 = 0$. We want to find some $x'_t \in \mathbb{F}_q \setminus \{x_1, \dots, x_t\}$ and $x \in \mathbb{F}_q \setminus \{x_1, \dots, x_{t-1}, x'_t\}$ such that $(x_1, \dots, x_{t-1}, x'_t, x)$ forms a solution of the following equation

$$\sum_{1 \leq i < j \leq t+1} X_i X_j = b.$$

Denote $a = x_1 + \cdots + x_t$. Because permutations of (x_1, \dots, x_t) are also solutions of the equation and $t \geq 3$, we may assume $x_t \neq a$. Thus

$$\begin{cases} a^2 - 2b &= \sum_{i=1}^t x_i^2, \\ a &= x_1 + \cdots + x_t. \end{cases}$$

It induces

$$\begin{cases} a^2 - 2x'_t(x_t - x'_t) - (x_t - x'_t)^2 + x^2 - 2b &= \sum_{i=1}^{t-1} x_i^2 + (x'_t)^2 + x^2, \\ a - (x_t - x'_t) + x &= x_1 + \cdots + x'_t + x. \end{cases}$$

To find solutions of the equation

$$\begin{cases} b &= \sum_{1 \leq i < j \leq t+1} X_i X_j, \\ X_i &\neq X_j, \quad \text{for all } i \neq j \end{cases}$$

in \mathbb{F}_q , it suffices to show the existences of $x'_t \in \mathbb{F}_q \setminus \{x_1, \dots, x_t\}$ and $x \in \mathbb{F}_q \setminus \{x_1, \dots, x_{t-1}, x'_t\}$ which satisfy

$$(a - (x_t - x'_t) + x)^2 = a^2 - 2x'_t(x_t - x'_t) - (x_t - x'_t)^2 + x^2.$$

That is, the solution

$$x = \frac{(x_t - x'_t)(x_t - a)}{x_t - x'_t - a} \in \mathbb{F}_q^*$$

satisfies

$$x \in \mathbb{F}_q^* \setminus \{x_2, \dots, x_{t-1}, x'_t\}.$$

If

$$x = \frac{(x_t - x'_t)(x_t - a)}{x_t - x'_t - a} = x_i, \quad \text{for some } i = 2, \dots, t-1,$$

then

$$x'_t = \frac{(x_t - x_i)(x_t - a)}{x_t - x_i - a}.$$

If

$$x = \frac{(x_t - x'_t)(x_t - a)}{x_t - x'_t - a} = x'_t,$$

then it has at most two solutions for x'_t , say c_1, c_2 . So, we can pick any element from

$$\mathbb{F}_q^* \setminus \{x_2, \dots, x_t, \frac{(x_t - x_2)(x_t - a)}{x_t - x_2 - a}, \dots, \frac{(x_t - x_{t-1})(x_t - a)}{x_t - x_{t-1} - a}, c_1, c_2\}$$

for x'_t . And under the condition

$$q - 1 > t - 1 + t - 2 + 2, \quad \text{i.e., } t \leq \frac{q-1}{2},$$

the equation

$$\begin{cases} b &= \sum_{1 \leq i < j \leq t+1} X_i X_j, \\ X_i &\neq X_j, \quad \text{for all } i \neq j, \end{cases}$$

always has solutions in \mathbb{F}_q . □

we calculate a sum of quadratic character of a finite field.

Lemma A.4. *Let q be an odd prime power and \mathbb{F}_q a finite field of q elements. For any $c \in \mathbb{F}_q$, we have*

$$\left| \sum_{x \in \mathbb{F}_q} \eta(x^2 + c) \right| = \begin{cases} q-1, & \text{if } c = 0, \\ 3, & \text{if } \eta(-1) = \eta(c) = -1, \\ 1, & \text{otherwise.} \end{cases}$$

where η is the quadratic character of \mathbb{F}_q , i.e.,

$$\eta(x) = \begin{cases} 1, & \text{if } x \text{ is a square,} \\ -1, & \text{if } x \text{ is not a square,} \\ 0, & \text{if } x = 0. \end{cases}$$

Proof. The statement holds obviously for $c = 0$.

Now we assume $c \neq 0$, consider the quadratic equation

$$x^2 + c = y^2,$$

namely,

$$(y - x)(y + x) = c.$$

It is easy to see that the equation has $(q-1)$ solutions

$$\left\{ \left(\frac{b - c/b}{2}, \frac{b + c/b}{2} \right) \mid b \in \mathbb{F}_q^* \right\}.$$

If $\eta(-1) = \eta(c) = -1$, then $-c$ is a square. So there are two solutions for the equation $x^2 + c = 0$, this means that

$$\left| \sum_{x \in \mathbb{F}_q} \eta(x^2 + c) \right| = \left| \frac{q-1-2}{2} - (q - \frac{q-1-2}{2}) + 0 + 0 \right| = 3.$$

For other cases, with the same argument, we finish the rests of this lemma. \square

By this lemma, we can compute the number of squares in a finite field whose images under a fixed affine map are still squares. It is interesting that squares are about one half elements in a finite field, and about one half squares are still squares under the action of an affine map.

Corollary A.5. *Let q be an odd prime power, and \mathbb{F}_q a finite field of q elements. For any $a, c \in \mathbb{F}_q^*$, we have*

$$A = \#\{x \in \mathbb{F}_q \mid \text{both } x \text{ and } ax + c \text{ are squares in } \mathbb{F}_q\} \geq \frac{q-1}{4}.$$

Proof. By Lemma A.4, we have

$$\left| \sum_{x \in \mathbb{F}_q} \eta(ax^2 + c) \right| = \left| \sum_{x \in \mathbb{F}_q} \eta(x^2 + ac) \right| = \begin{cases} 3, & \text{if } \eta(-1) = \eta(ac) = -1, \\ 1, & \text{otherwise.} \end{cases}$$

Let $m = \#\{x \in \mathbb{F}_q \mid ax^2 + c \text{ is a square in } \mathbb{F}_q\}$, then

$$\left| \sum_{x \in \mathbb{F}_q} \eta(ax^2 + c) \right| = \begin{cases} |m - 2 - (q - m)|, & \text{if } \eta(-ac) = -1, \\ |m - (q - m)|, & \text{if } \eta(-ac) \neq -1. \end{cases}$$

Comparing the above two equalities, it follows that

$$m \geq \frac{q-1}{2}.$$

Since $A = \frac{m+(\eta(c)+1)/2}{2} \geq \frac{m}{2}$, we complete the proof. \square

If we only consider $\frac{q}{7} < t \leq \frac{q-1}{2}$, taking $c = 1/2$ and $\epsilon = \sqrt{2} \left(\frac{1}{q^{14/q}} - \frac{1}{p} - \frac{1}{2} \right)$ in the Theorem 5.3 in [7] where p is the characteristic of the finite field \mathbb{F}_q , then we have

Proposition A.6. *If $q \geq 257$ and $\frac{q}{7} < t \leq \frac{q-1}{2}$, then for any $a, b \in \mathbb{F}_q$, the system of equations*

$$\begin{cases} a &= \sum_{i=1}^t X_i, \\ b &= \sum_{1 \leq i < j \leq t} X_i X_j, \\ X_i &\neq X_j, \quad \text{for all } i \neq j \end{cases}$$

has solutions in \mathbb{F}_q .

Theorem A.7. *If $q \geq 257$ and $\frac{q}{7} < t < \frac{6q}{7}$, then for any $a, b \in \mathbb{F}_q$, the system of equations*

$$\begin{cases} a &= \sum_{i=1}^t X_i, \\ b &= \sum_{1 \leq i < j \leq t} X_i X_j, \\ X_i &\neq X_j, \quad \text{for all } i \neq j \end{cases}$$

has solutions in \mathbb{F}_q .

Proof. The statement is equivalent to that there exist pairwise distinct $\alpha_1, \dots, \alpha_t \in \mathbb{F}_q$ such that

$$1 + ax + bx^2 \equiv \prod_{j=1}^t (1 - \alpha_j x) \pmod{x^3}.$$

By Proposition A.6, it suffices to prove that if $\frac{q+1}{2} \leq t < \frac{6q}{7}$, for any $a, b \in \mathbb{F}_q$, there exist pairwise distinct $\alpha_1, \dots, \alpha_t \in \mathbb{F}_q$ such that

$$1 + ax + bx^2 \equiv \prod_{j=1}^t (1 - \alpha_j x) \pmod{x^3}.$$

So it is equivalent to

$$\frac{1 - x^{q-1}}{1 + ax + bx^2} \equiv \frac{1 - x^{q-1}}{\prod_{j=1}^t (1 - \alpha_j x)} \pmod{x^3}.$$

Set

$$\{\beta_1, \dots, \beta_{q-t}\} = \mathbb{F}_q \setminus \{\alpha_1, \dots, \alpha_t\}.$$

Then it is equivalent to the existence of distinct $\beta_1, \dots, \beta_{q-t} \in \mathbb{F}_q$ satisfying

$$\frac{1 - x^{q-1}}{1 + ax + bx^2} \equiv \prod_{j=1}^{q-t} (1 - \beta_j x) \pmod{x^3}.$$

By Proposition A.6, when $\frac{q}{7} < q - t \leq \frac{q-1}{2}$, i.e.,

$$\frac{q+1}{2} \leq t < \frac{6q}{7},$$

the required $\beta_1, \dots, \beta_{q-t}$ exist. □

Now we prove Theorem 4.2.

Proof of Theorem 4.2.

(a) By Lemmas A.1-A.3, when $\frac{q+1}{2} \leq t \leq q-2$, for any $a, b \in \mathbb{F}_q$, the equation

$$\begin{cases} b = \sum_{1 \leq i < j \leq t} X_i X_j + \sum_{i=1}^t X_i^2 - a \sum_{i=1}^t X_i, \\ X_i \neq X_j, \quad \text{for all } i \neq j \end{cases}$$

has solutions in \mathbb{F}_q .

(b) For $3 \leq t < \frac{q-11}{6}$, we prove the statement by induction on t .

If $t = 3$, first we fix an element $x_1 \in \mathbb{F}_q$ such that

$$2x_1^2 - 3b \neq 0.$$

Now we need to find $x_2 \in \mathbb{F}_q \setminus \{x_1\}$ such that the equation on X

$$X^2 + (x_1 + x_2)X + x_1^2 + x_2^2 + x_1 x_2 - b = 0$$

has a solution $x_3 \in \mathbb{F}_q \setminus \{x_1, x_2, -x_1 - x_2\}$.

It is well-known that the above equation on X has solutions if and only if the discriminant

$$\begin{aligned} \Delta &= (x_1 + x_2)^2 - 4(x_1^2 + x_2^2 + x_1 x_2 - b) \\ &= -3x_2^2 - 2x_1 x_2 - 3x_1^2 + 4b \end{aligned}$$

is a square in \mathbb{F}_q .

Note that the characteristic $p \neq 3$, so the discriminant

$$\Delta = -3(x_2 + \frac{x_1}{3})^2 - \frac{8x_1^2}{3} + 4b.$$

Denote

$$x_2(\Delta) = \{x_2 \in \mathbb{F}_q \setminus \{x_1\} \mid \text{the induced } \Delta \text{ is a square}\}.$$

Note that $x_2 \neq x_1$, by Corollary A.5, we have

$$|x_2(\Delta)| \geq \left(2 \left(\frac{q-1}{4} - 1\right) + 1\right) - 1 = \frac{q-5}{2}.$$

Since the equation on x_2

$$X^2 + (x_1 + x_2)X + x_1^2 + x_2^2 + x_1x_2 - b = 0$$

gives at most two solutions $x_i^{(1)}, x_i^{(2)}$ for each $x = x_i$ ($i = 1, 2$) or $c^{(1)}, c^{(2)}$ for $x = -x_1 - x_2$, respectively.

From the argument above, we can pick any element from the set

$$x_2(\Delta) \setminus \{x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)}, c^{(1)}, c^{(2)}\}$$

for x_2 . And this can be done under the assumption

$$6 < \frac{q-5}{2}, \quad \text{i.e., } q > 17.$$

Now we assume that the result is true for $t \geq 3$.

Then for $t+1$, by induction hypothesis, we may assume $(x_1, \dots, x_t) \in \mathbb{F}_q^t$ forms a solution of the equation

$$\begin{cases} b = \sum_{1 \leq i < j \leq t} X_i X_j + \sum_{i=1}^t X_i^2, \\ X_i \neq X_j, \quad \text{for all } i \neq j, \end{cases}$$

and

$$a = \sum_{i=1}^t x_i \neq 0.$$

Then

$$\begin{cases} 2b - a^2 = \sum_{i=1}^t x_i^2, \\ a = \sum_{i=1}^t x_i. \end{cases}$$

The same as the proof in Lemma A.3, if $x_i \neq 0$ for all $1 \leq i \leq t$, then $(x_1, \dots, x_t, 0) \in \mathbb{F}_q^t$ forms a solution of the equation

$$\begin{cases} b = \sum_{1 \leq i < j \leq t+1} X_i X_j + \sum_{i=1}^{t+1} X_i^2, \\ X_i \neq X_j, \quad \text{for all } i \neq j, \end{cases}$$

Otherwise, assume $x_t = 0$. We replace x_t by a proper element $x'_t \in \mathbb{F}_q^*$ to induce an element $x \in \mathbb{F}_q$ such that $(x_1, \dots, x'_t, x) \in \mathbb{F}_q^{t+1}$ is a solution of the equation

$$\begin{cases} b = \sum_{1 \leq i < j \leq t+1} X_i X_j + \sum_{i=1}^{t+1} X_i^2, \\ X_i \neq X_j, \quad \text{for all } i \neq j, \end{cases}$$

and

$$\sum_{i=1}^{t-1} x_i + x'_t + x \neq 0.$$

Firstly, we pick $x'_t \in \mathbb{F}_q \setminus \{x_1, \dots, x_t\}$, which will be determined. Then by the induction hypothesis,

$$\begin{cases} 2b - a^2 + (x'_t)^2 + x^2 &= \sum_{i=1}^{t-1} x_i^2 + (x'_t)^2 + x^2, \\ a + x'_t + x &= x_1 + \dots + x'_t + x. \end{cases}$$

We want to find some $x'_t \in \mathbb{F}_q \setminus \{x_1, \dots, x_t, -a\}$ such that the equation on x

$$(a + x'_t + x)^2 = a^2 - (x'_t)^2 - x^2$$

gives a solution $x \in \mathbb{F}_q \setminus \{x_1, \dots, x'_t, -(\sum_{i=1}^{t-1} x_i + x'_t)\}$. The equation is

$$x^2 + (a + x'_t)x + x'_t(a + x'_t) = 0.$$

Here, we see that why we require $x'_t \neq -a$ at the beginning. Indeed, if $x'_t = -a$, then the equation above is reduced to

$$x^2 = 0,$$

which has only zero solution $x = 0$. Then

$$\sum_{i=1}^{t-1} x_i + x'_t + x = 0,$$

which goes against our requirement in the induction.

Similarly as the proof for the case $t = 3$, the above equation on x has solutions if and only if the discriminant

$$\begin{aligned} \Delta &= (a + x'_t)^2 - 4x'_t(a + x'_t) \\ &= -3(x'_t)^2 - 2ax'_t + a^2 \end{aligned}$$

is a square in \mathbb{F}_q .

Note that the characteristic $p \neq 3$, so the discriminant

$$\Delta = -3(x'_t + \frac{a}{3})^2 + \frac{4a^2}{3}.$$

Denote

$$x'_t(\Delta) = \{x'_t \in \mathbb{F}_q^* \mid \text{the induced } \Delta \text{ is a square}\}.$$

Note that $x'_t \neq 0$, by Corollary A.5, we have

$$|x'_t(\Delta)| \geq \left(2 \left(\frac{q-1}{4} - 1\right) + 1\right) - 1 = \frac{q-5}{2}.$$

Since the equation on x'_t

$$x^2 + (a + x'_t)x + x'_t(a + x'_t) = 0$$

gives at most two solutions $x_i^{(1)}, x_i^{(2)}$ for each $x = x_i$ ($i = 1, 2, \dots, t-1$) or $c^{(1)}, c^{(2)}$ for $x = -(\sum_{i=1}^{t-1} x_i + x'_t)$ or $(x'_t)^{(1)}, (x'_t)^{(2)}$ for $x = x'_t$, respectively.

From the argument above, we pick any element x'_t from the set

$$x'_t(\Delta) \setminus \{x_1, \dots, x_t, x_1^{(1)}, x_1^{(2)}, \dots, x_{t-1}^{(1)}, x_{t-1}^{(2)}, (x'_t)^{(1)}, (x'_t)^{(2)}, c^{(1)}, c^{(2)}, -a\}$$

and finish the induction. And under the assumption

$$3t + 3 < \frac{q-5}{2}, \quad \text{i.e., } t < \frac{q-11}{6},$$

such an x'_t always exists.

Since

$$\frac{q}{7} < \frac{q-11}{6}$$

as $q \geq 257$, by combining (a), (b) and Theorem A.7, we complete the proof of Theorem 4.2. \square